

Gravitomagnetic effect and spin-torsion coupling

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February 7, 2008

Abstract

We study the gravitomagnetic effect in the context of absolute parallelism with the use of a modified geodesic equation via a free parameter b . We calculate the time difference in two atomic clocks orbiting the Earth in opposite directions and find a small correction due the coupling between the torsion of space time and the internal structure of atomic clocks measured by the free parameter.

PACS NUMBERS: 04.80.Cc, 04.90.+e

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1 Introduction

In attempting to seek experimental confirmation of the gravitomagnetic effect suggested by Mashhoon *et al.* [1], the called Gravity Probe C(lock)

experiment was proposed [2]. The experimental confirmation is considered difficult due several perturbations of planetary origin that can hide the gravitomagnetic effect [3]. In the experiment a clock is sent in a direct equatorial and circular orbit, and another clock in a retrograde orbit, both clocks considered without internal structure. The time difference marked by the clocks is expected to be $(4\pi a)/c \sim 2,327 \times 10^{-7}s$. This difference is considered for an exterior observer with $r \gg (2GM)/c^2$, where $a = J/Mc$, J is the Earth angular momentum, M is the mass of the Earth, G is the Gravitational constant and c is the speed of light. This effect is interpreted as the dragging of a inertial frame due to the Earth's rotation.

It is well-known that photon's trajectories are of fundamental importance for astronomy in several observed waves lengths. Photons move in space-time according to the geodesic equation, where the Christoffel connection plays a fundamental role. This geodesic equation embodies Einstein's equivalence principle. The trajectories of the photons in the space-time are used in the explanation of the following classical tests of general relativity: the test of the redshift, the light rays deflection and the time delay of radar signals around planets, known as Shapiro effect. In these tests, the photons are treated as light rays, that is, particles without spin. Investigations carried out with the help of the parametrized post-Newtonian formalism (PPN) suggest several observations to prove these tests. They produced results that prove the predictions of general relativity with high precision [4]. However, although the agreement favours general relativity, it does not mean that some corrections to the theory cannot be implemented, corrections that yield results which agree with the experimental error limits. Thus it is possible that the general relativity theory be a correct gravitational theory within certain limits.

Wanas and Kahil [5] and Wanas *et al.* [6] proposed to explain the discrepancy between the thermal neutrons interference experiment and the theoretical prediction, by means of Bazanski's formalism [7], through the "quantization" of the path followed by the particles with spin. They used a modified geodesic equation to include Einstein's absolute parallelism using a nonsymmetrical connection. They applied this equation to the weak field limit and found that the Newtonian gravitational potential is modified for a factor $(1 - b)$, where b establishes the coupling between the torsion field and the intrinsic spin. For particles with spin, they postulate that $b = (n/2)\alpha\gamma$, where α is the fine structure constant and γ is a parameter to match with the experience. For $n = 0, 1, 2, 3, \dots$ the particles assume spin *zero*, $1/2$, 1 , $3/2$ etc. For

macroscopic bodies (without spin), $n = 0$ and $b = 0$. This interaction would take place through the coupling of the spin particle with the space-time torsion. However, new experiments would need to be accomplished to test such quantization of the path.

In this article we propose a new test to verify the gravitomagnetic effect and Wanas' conclusions by considering a covariant derivative definition $D_\mu e_{a\nu} = 0$ in the absolute parallelism framework, that yields a class of geodesic equations, and taking into account the identity $\omega_{\mu ab} = {}^0\omega_{\mu ab} + K_{\mu ab}$, where $\omega_{\mu ab}$ is an arbitrary affine connection, $e_{a\mu}$ is tetrad field with Lorentz indices a, b, \dots , ${}^0\omega_{\mu ab}$ is the Levi-Civita connection and $K_{\mu ab}$ is the contortion tensor. We impose the time gauge condition [8] for the tetrad field by fixing $e_{(k)}^0 = 0$ and $e^{(0)}_k = 0$. We find the same geodesic equation obtained by Wanas by assuming that for particles with nonzero spin the violation of the equivalence principle is negligible, and therefore the coupling with the torsion is very small. The latter takes place by means of an empirical parameter b , that characterizes the coupling between torsion and the spin of particles. When applying this new equation to the Kerr metric, for a circular and equatorial orbit, we find that the period difference measured by the clocks is about $[(4\pi a)/c][(1 - 2b)/(1 - b)]$. We also conclude that the orbital period is given by $T_o = \frac{2\pi}{\omega_0(1-b)^{1/2}}$, which is larger than the expected value and indicates that the Newtonian gravitational potential is modified by means of a factor $(1 - b)$, namely, $\phi = -\frac{GM}{r}(1 - b)$. Therefore there is a modification of order $(1 - b)^{1/2}$ in the Keplerian period. The reason for this is that the potential on the clock must be smaller than the usual Newtonian potential by a factor $(1 - b)$. The clock would be under the action of a smaller potential, with a smaller acceleration, registering a longer time to complete an orbit. When considering clocks without internal structure, as done previously, we make $b = 0$. Due to the fact that measurements of time differences of such low order require the use atomic clocks, as H maser (maser of Hydrogen) and Cesium 133, we suggest a coupling of the internal structure of these clocks with the space-time torsion. The frequency of the Cs atomic clock is $\nu_0 \approx 9.2$ GHz and corresponds to the $[F = 4, m_F = 0] \rightarrow [F = 3, m_F = 0]$ hyperfine transition in the ^{133}Cs ground state. As for the H maser there corresponds the $[F = 1, m_F = 0] \rightarrow [F = 0, m_F = 0]$ transition with frequency $\nu_0 \approx 1.4$ GHz. The violation of local position invariance incorporated in Einstein's equivalence principle can be used to quantify the dimensionless

parameter β (positive or negative) that measures the discrepancy between the observed and predicted redshift $\Delta\nu$ of spectral lines of atomic clocks (see, for example, [4] and references therein). The parameter β depends on the nature of the measured clock. The parameter b can be determined in the same manner by means of the expression $\Delta\nu/\nu_0 = [1 + (1 - b)\phi_N/c^2]$, where $\phi_N = -GM/r$ is the Newtonian gravitational potential. For two identical ^{133}Cs clocks the result is $|b| < 1.5 \times 10^{-2}$. Consequently this value represent a 1.52% difference with respect to Mashhoon's prediction.

We have considered the weak field approximation of modified geodesic equations that satisfy the Newtonian limit in an arbitrary teleparallel theory. The latter theory is defined to be quadratic in the torsion tensor with free parameters c_1 , c_2 and c_3 [9], [10]. The condition of Legendre transform for a well defined Hamiltonian formulation is given by $c_1 + c_2 = 0$. In the present context we found that $c_1 = -\frac{2}{3}k\frac{1}{(1-b)}$, $c_2 = \frac{2}{3}k\frac{1}{(1-b)}$ and $c_3 = -\frac{3}{2}k\frac{1}{(1-b)}$, where $k = \frac{c^3}{16\pi G}$, indicating that if we consider particles without internal structure (spin), then $b = 0$, resulting in the teleparallel equivalent of general relativity. Thus b cannot be 1, what it is agreement with the fact that the torsion coupling with spin must be small, resulting in a small violation of the principle of equivalence.

In section 2 we review the geodesic equation for particles in a gravitational field with a symmetric connection. In section 3, we introduce the tetrad field description of the Weitzenböck space-time. In section 4 we carry out the calculations of the gravitomagnetic effect with a nonsymmetric connection via the free empirical parameter b , displaying the difference with respect to general relativity. In the section 5 we provide estimates of the empirical parameter b . In section 6, we introduce the relationship with the teleparallel equivalent of general relativity in the weak field approximation. In section 7 the conclusions are presented.

The notation is the following: space-time indices μ, ν, \dots and $SO(3, 1)$ Lorentz indices a, b, \dots run from 0 to 3. In the 3+1 decomposition Latin indices from the middle of the alphabet indicate space indices according to $\mu = 0, i$ and $a = (0), (i)$. The flat space-time metric is fixed by $\eta_{(0)(0)} = -1$.

2 The gravitomagnetic effect in the general relativity

The exterior space-time of a system with mass M and specific angular momentum $a = J/M$ is described by the Kerr geometry. The Kerr metric is an exact solution of the vacuum field equations of general relativity. Written in Boyer-Lindquist coordinates (t, r, θ, φ) , the Kerr metric reads

$$ds^2 = -dt^2 + \Sigma \left(\frac{1}{\Delta} dr^2 + d\theta^2 \right) + (r^2 + a^2) \sin^2 \theta d\varphi^2 + 2M \frac{r}{\Sigma} (dt - a \sin^2 \theta)^2, \quad (1)$$

where we have $\Sigma = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$, and in this section $G = c = 1$.

We first calculate the time registered by a standard clock that follows a geodesic in the Kerr geometry [1]. We choose a circular and equatorial orbit. The geodesic equation results in

$$dt^2 - 2ad\varphi dt + \left(a^2 - \frac{r^3}{M} \right) d\varphi^2 = 0, \quad (2)$$

whose solutions are

$$\frac{dt}{d\varphi} = a \pm \left(\frac{r^3}{M} \right)^{1/2} = a \pm \frac{1}{\omega_0}, \quad (3)$$

where ω_0 is the keplerian angular velocity.

We find, with the help of (1), the relation

$$\left(\frac{d\tau}{d\varphi} \right)^2 = \left(1 - \frac{2M}{r} \right) \left(\frac{dt}{d\varphi} \right)^2 + 4 \frac{Ma}{r} \frac{dt}{d\varphi} - r^2 - a^2 \left(1 + 2 \frac{M}{r} \right). \quad (4)$$

Substituting (3) in the equation above we obtain, for a closed orbit, and considering an observer at the infinity such that $r \gg 2M$,

$$\tau_+ - \tau_- \approx 4\pi a = 4\pi \frac{J}{M}, \quad (5)$$

where the signals $+$ and $-$ apply for a direct and retrograde orbit, respectively.

Introducing the speed of light c ,

$$\begin{aligned}\tau_+ - \tau_- &\approx 4\pi \frac{J}{Mc^2} \\ &\approx 2,327 \times 10^{-7} s,\end{aligned}\tag{6}$$

where we used $M \approx 6 \times 10^{24} \text{ kg}$ and $J \approx 10^{34} \text{ kg.m}^2.\text{s}^{-1}$, the mass and angular momentum of the Earth, respectively. We can now ask what would happen if the clock would follow a geodesic different from the Riemannian one, for example, one due to a nonsymmetrical connection.

3 The Weitzenböck space-time

We present now a brief summary of the space-time of the Riemann-Cartan type that is endowed with a metric $g_{\mu\nu}$ and a connection $\Gamma_{\mu\nu}^\lambda$.

The Riemann-Cartan space-time is characterized by [11]

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma_{\mu\lambda}^\rho g_{\rho\nu} - \Gamma_{\nu\lambda}^\rho g_{\mu\rho} = 0.\tag{7}$$

From this equation we obtain

$$\Gamma_{\mu\nu}^\lambda = {}^0\Gamma_{\mu\nu}^\lambda + K_{\mu\nu}^\lambda,\tag{8}$$

where the first member on the right hand side is the Christoffel connection,

$${}^0\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}),\tag{9}$$

and second term is the contortion tensor,

$$K_{\mu\nu}^\lambda = \frac{1}{2} (T_{\mu\nu}^\lambda + T_\mu^{\lambda\ \nu} - T_\nu^{\lambda\ \mu}).\tag{10}$$

The torsion tensor is given by

$$T_{\mu\nu}^\lambda(\Gamma) = \Gamma_{\mu\nu}^\lambda - \Gamma_{\nu\mu}^\lambda,\tag{11}$$

and the curvature tensor by

$$R^\mu_{\ \nu\alpha\beta}(\Gamma) = \partial_\alpha \Gamma_{\beta\nu}^\mu - \partial_\beta \Gamma_{\alpha\nu}^\mu + \Gamma_{\alpha\sigma}^\mu \Gamma_{\beta\nu}^\sigma - \Gamma_{\beta\sigma}^\mu \Gamma_{\alpha\nu}^\sigma.\tag{12}$$

The Riemann-Cartan space-time is characterized by nonzero curvature and torsion tensors. It leads to two geometrical models for the space-time. The first is the Riemannian space-time, that is obtained by requiring the vanishing of the torsion tensor. Therefore the space-time affine connection reduces to the Christoffel connection. Another model is the Weitzenböck space-time, that is obtained from Riemann-Cartan space-time by requiring the curvature tensor to vanish,

$$R^\mu{}_{\nu\alpha\beta}(\Gamma) = 0. \quad (13)$$

The Weitzenböck space-time is endowed with the affine connection

$$\Gamma^\lambda_{\mu\nu} = e_a{}^\lambda \partial_\mu e^a{}_\nu = -e^a{}_\nu \partial_\mu e_a{}^\lambda. \quad (14)$$

where $e^a{}_\mu$ are orthonormal tetrads. The indices a, b, c, \dots are called local tetrads or indices of the $SO(3, 1)$ group.

The affine connection (14) is not symmetrical with respect to a change of the lower indices. Therefore the torsion tensor is given by

$$\begin{aligned} T^\lambda{}_{\mu\nu}(\Gamma) &= \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = \\ &= e_a{}^\lambda (\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu). \end{aligned} \quad (15)$$

From now on we will adopt the Weitzenböck space-time. In other words, the space-time will be characterized by $R^\rho{}_{\sigma\mu\nu}(\Gamma) = 0$ and $T^\lambda{}_{\mu\nu}(\Gamma) \neq 0$.

4 Gravitomagnetic effect with a nonsymmetrical connection

In the Weitzenböck space-time the covariant derivative of the tetrad field $e_{a\mu}$ vanish,

$$D_\mu e_{a\nu} = 0, \quad (16)$$

from what follows

$$e^{a\lambda} \partial_\mu e_{a\nu} = {}^0\Gamma^\lambda_{\mu\nu} - e^{a\lambda} e^b{}_\nu {}^0\omega_{\mu ab}, \quad (17)$$

where ${}^0\omega_{\mu ab}$ is the Levi-Civita connection, which plays an important role in the interaction of spin 1/2 matter fields with the gravitational field. For an arbitrary connection $\omega_{\mu ab}$ there exists the identity

$$\omega_{\mu ab} = {}^0\omega_{\mu ab} + K_{\mu ab}, \quad (18)$$

where $K_{\mu ab}$ is the contortion tensor. It follows that by fixing $\omega_{\mu ab} = 0$, we obtain

$${}^0\omega_{\mu ab} = -K_{\mu ab} = -\frac{1}{2}e_a{}^\lambda e_b{}^\nu (T_{\lambda\mu\nu} + T_{\nu\lambda\mu} - T_{\mu\nu\lambda}), \quad (19)$$

and

$$e^{a\lambda}e^b{}_\nu {}^0\omega_{\mu ab} = \frac{1}{2}g^{\lambda\rho} (T_{\mu\nu\rho} + T_{\nu\mu\rho}) - \frac{1}{2}g^{\lambda\rho}T_{\rho\mu\nu}, \quad (20)$$

which, except for the parameter b , allows us to rewrite equation (17) as

$$\Gamma_{\mu\nu}^\lambda = e^{a\lambda}\partial_\mu e_{a\nu} = {}^0\Gamma_{\mu\nu}^\lambda - b \left[\frac{1}{2}g^{\lambda\rho} (T_{\mu\nu\rho} + T_{\nu\mu\rho}) + \frac{1}{2}g^{\lambda\rho}T_{\rho\mu\nu} \right]. \quad (21)$$

The empirical parameter b has been introduced to account for observational or experimental evidences. For $b = 1$, the connection (21) reduces to Cartan's connection, describing the autoparallels (the straightest curves in Riemann-Cartan space) [12]. For $b = 0$, we recover the Christoffel connection together with the results of section 2.

The fixation of $\omega_{\mu ab} = 0$ seems to be important for a well defined Hamiltonian formulation, and in order to have a correct time evolution of the field quantities in the realm of the teleparallel equivalent to the general relativity (TEGR) [10], [13], [14].

Assuming that we can test a new geodesic equation by substituting the Christoffel connection by the nonsymmetrical connection (21), we can write

$$\frac{d^2x^\lambda}{d\tau^2} + {}^0\Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - b \frac{g^{\lambda\rho}}{2} (T_{\mu\nu\rho} + T_{\nu\mu\rho}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \quad (22)$$

The equation above is identical to the one found by Wanas and Kahil [5] using a variational principle in the context of Bazanski's formalism [7] for the space-time of absolute parallelism.

By using equation (22) we are going to calculate the time difference measured by a clock in direct orbit around the Earth and by another one in retrograde orbit, and eventually we will compare the result with that of section 2. To this purpose, we are going to use the line element

$$ds^2 = -\frac{\Delta}{\rho^2} (cdt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\varphi - a cdt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (23)$$

The metric in spherical coordinates is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{\Psi^2}{\rho^2} & 0 & 0 & -\frac{\chi \sin^2 \theta}{\rho^2} \\ 0 & \frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & \rho^2 & 0 \\ -\frac{\chi \sin^2 \theta}{\rho^2} & 0 & 0 & \frac{\Sigma^2 \sin^2 \theta}{\rho^2} \end{pmatrix}. \quad (24)$$

We also have

$$g^{\mu\nu} = \begin{pmatrix} \frac{-\rho^2 \Sigma^2}{\Psi^2 \Sigma^2 + \chi^2 \sin^2 \theta} & 0 & 0 & \frac{-\rho^2 \chi}{\Psi^2 \Sigma^2 + \chi^2 \sin^2 \theta} \\ 0 & \frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & \frac{1}{\rho^2} & 0 \\ \frac{-\rho^2 \chi}{\Psi^2 \Sigma^2 + \chi^2 \sin^2 \theta} & 0 & 0 & \frac{\rho^2 \Psi^2}{(\Psi^2 \Sigma^2 + \chi^2 \sin^2 \theta) \sin^2 \theta} \end{pmatrix}, \quad (25)$$

with the following definitions

$$\Delta = r^2 + a^2 - 2\frac{GM}{c^2}r, \quad (26)$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad (27)$$

$$\Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \quad (28)$$

$$\Psi^2 = \Delta - a^2 \sin^2 \theta, \quad (29)$$

$$\chi = 2a\frac{GM}{c^2}r. \quad (30)$$

With the purpose of simplifying the calculations, we consider a circular and equatorial orbit $r = 1 = \text{constant}$, and $\theta = \frac{\pi}{2}$. Thus, using equations (22) and (25) we find, after a long calculation,

$$\begin{aligned}
& c^2 dt^2 + 2 \frac{{}_0\Gamma_{03}^r}{{}_0\Gamma_{00}^r} c dt d\varphi + \\
& + \frac{{}_0\Gamma_{33}^r}{{}_0\Gamma_{00}^r} d\varphi^2 - b \frac{1}{{}_0\Gamma_{00}^r} [g^{11} e^{(0)} {}_0T_{(0)01} dx^0 dx^0 + \\
& + g^{11} e^{(1)} {}_3T_{(1)01} dx^3 dx^0 + \\
& + g^{11} e^{(2)} {}_3T_{(2)01} dx^3 dx^0 + \\
& + g^{11} e^{(1)} {}_3T_{(1)31} dx^3 dx^3 + \\
& + g^{11} e^{(2)} {}_3T_{(2)31} dx^3 dx^3 + \\
& + g^{11} e^{(3)} {}_3T_{(3)31} dx^3 dx^3 + \\
& + g^{11} e^{(1)} {}_0T_{(1)01} dx^0 dx^0 + \\
& + g^{11} e^{(2)} {}_0T_{(2)01} dx^0 dx^0 + \\
& + g^{11} e^{(2)} {}_0T_{(2)31} dx^0 dx^3 + \\
& + g^{11} e^{(1)} {}_0T_{(1)31} dx^0 dx^3] \\
& = 0.
\end{aligned} \tag{31}$$

We have adopted Schwinger's time gauge [8],

$$e^{(0)}{}_i = e_{(0)i} = 0, \quad e^{(k)0} = 0. \tag{32}$$

In the case of asymptotically flat space-times, the tetrad fields that satisfy Schwinger's time gauge condition, and the symmetric condition in Cartesian coordinates [15, 16],

$$e_{(i)j}(t, x, y, z) = e_{(j)i}(t, x, y, z), \tag{33}$$

are given by,

$$e_{a\mu} = \begin{pmatrix} -\frac{1}{\rho} \sqrt{\Psi^2 + \frac{\chi^2}{\Sigma^2} \sin^2 \theta} & 0 & 0 & 0 \\ \frac{\chi}{\Sigma \rho} \sin \theta \sin \varphi & \frac{\rho}{\sqrt{\Delta}} \sin \theta \cos \varphi & \rho \cos \theta \cos \varphi & -\frac{\Sigma}{\rho} \sin \theta \sin \varphi \\ -\frac{\chi}{\Sigma \rho} \sin \theta \cos \varphi & \frac{\rho}{\sqrt{\Delta}} \sin \theta \sin \varphi & \rho \cos \theta \sin \varphi & \frac{\Sigma}{\rho} \sin \theta \cos \varphi \\ 0 & \frac{\rho}{\sqrt{\Delta}} \cos \theta & -\rho \sin \theta & 0 \end{pmatrix}. \tag{34}$$

Certainly there is a infinity of tetrads that yield the metric tensor (24), but only one that leads to the correct gravitational energy description [15].

Considering that

$${}^0\Gamma_{00}^r = -\frac{1}{2}g^{11}\partial_r g_{00} = \frac{GM}{c^2 r^4} \left(r^2 + a^2 - 2\frac{GM}{c^2}r \right), \quad (35)$$

$$\frac{{}^0\Gamma_{03}^r}{{}^0\Gamma_{00}^r} = \frac{\partial_r g_{03}}{\partial_r g_{00}} = -2a, \quad (36)$$

$$\frac{{}^0\Gamma_{33}^r}{{}^0\Gamma_{00}^r} = \frac{\partial_r g_{33}}{\partial_r g_{00}} = \left(a^2 - \frac{r^3 c^2}{GM} \right), \quad (37)$$

and

$$T_{(0)01} = \frac{1}{2} \left[1 - \frac{2MG/c^2}{r} + \frac{4a^2 G^2 M^2 / c^4}{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2 \right]} \right]^{-1/2} \times \\ \times \left[\left(\frac{2MG/c^2}{r^2} - \frac{4a^2 G^2 M^2 / c^4 [4(r^2 + a^2)r - 2(r - GM/c^2)a^2]}{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2 \right]^2} \right) \right], \quad (38)$$

$$T_{(1)01} = \left[\frac{2a \frac{GM}{c^2} [4(r^2 + a^2)r - 2(r - \frac{GM}{c^2})a^2]}{(r^2 + a^2)^2 - (r^2 + a^2 - 2\frac{GM}{c^2}r)a^2} \right] \sin \phi, \quad (39)$$

$$T_{(2)01} = \left[\frac{2a \frac{GM}{c^2} [4(r^2 + a^2)r - 2(r - \frac{GM}{c^2})a^2]}{(r^2 + a^2)^2 - (r^2 + a^2 - 2\frac{GM}{c^2}r)a^2} \right] \cos \phi, \quad (40)$$

$$T_{(1)31} = -\frac{r \sin \varphi}{\left(r^2 + a^2 - 2\frac{GM}{c^2}r \right)^{1/2}} + \\ + \left\{ \frac{2(r^2 + a^2)r - \left(r - \frac{GM}{c^2} \right) a^2}{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2 \right]^{1/2}} \right\} \frac{\sin \varphi}{r} -$$

$$-\left\{\frac{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2\right]^{1/2}}{r^2}\right\} \sin \varphi, \quad (41)$$

$$\begin{aligned} T_{(2)31} &= \frac{r \cos \varphi}{\left(r^2 + a^2 - 2\frac{GM}{c^2}r\right)^{1/2}} - \\ &-\left\{\frac{2(r^2 + a^2)r - \left(r - \frac{GM}{c^2}\right)a^2}{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2\right]^{1/2}}\right\} \frac{\cos \varphi}{r} + \\ &+\left\{\frac{\left[(r^2 + a^2)^2 - (r^2 + a^2 - 2GMr/c^2) a^2\right]^{1/2}}{r^2}\right\} \cos \varphi, \end{aligned} \quad (42)$$

$$T_{(3)31} = 0, \quad (43)$$

we can rewrite equation (31), in the form

$$a'c^2 dt^2 + b'cdtd\varphi + c'd\varphi^2 = 0, \quad (44)$$

where

$$a' = 1 - b, \quad (45)$$

$$b' = -2a \left\{ 1 - b \left[1 + \frac{r^2}{\left[\left(r^2 + a^2 - 2\frac{GM}{c^2}r\right) \left(r^2 + a^2 + 2\frac{GM}{c^2}\frac{a^2}{r}\right)\right]^{1/2}} \right] \right\}, \quad (46)$$

$$c' = \left\{ a^2 - \frac{c^2}{\omega_0^2} \left[1 + b \left(1 - \frac{GMa^2}{c^2 r^3} - \left[\frac{\left(r^2 + a^2 + 2\frac{GM}{c^2}\frac{a^2}{r}\right)}{\left(r^2 + a^2 - 2\frac{GM}{c^2}r\right)} \right]^{1/2} \right) \right] \right\}, \quad (47)$$

and $\omega_0^2 = \frac{GM}{r^3}$.

The square time interval $d\tau^2$ is calculated by means of the line element (23),

$$\begin{aligned} \left(\frac{d\tau}{d\varphi}\right)^2 &= \left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{d\varphi}\right)^2 + \frac{1}{c} \frac{dt}{d\varphi} \left(\frac{4GMa}{c^2 r}\right) - \\ &\quad - \frac{1}{c^2} a^2 \left(1 + \frac{2GM}{c^2 r}\right) - \frac{r^2}{c^2}. \end{aligned} \quad (48)$$

With the help of expression (44) we can write

$$c \frac{dt}{d\varphi} = \frac{-b' \pm (b'^2 - 4a'c')^{1/2}}{2a'}. \quad (49)$$

Substituting the equation above in expression (48) and integrating in φ from 0 to 2π , we find the square time differences

$$\frac{\tau_+^2 - \tau_-^2}{2T_0} \sim \frac{4\pi a}{c} \frac{(1 - 2b)}{(1 - b)^{3/2}}, \quad (50)$$

in the limit $r \gg 2GM/c^2$ and $r \gg a$. $T_0 = \frac{2\pi}{\omega_0(1-b)^{1/2}}$ is the orbital period. For $b = 0$ the expression (50) coincides with the one found by Mashhoon *et al.* [1], who considered clocks as point particles, i.e., without internal structure.

For $b = 1$ the first term containing $(1 - b)c^2 dt^2$ in equation (44) vanishes, a fact that prevents from recovering the general relativity limit. By keeping $b \neq 1$ in the limit $r \gg 2GM/c^2$ and $r \gg a$, we obtain the periods for a direct and retrograde orbit

$$\tau_{\pm} \sim \frac{2\pi}{\omega_0(1-b)^{1/2}} \pm \frac{2\pi a}{c} \frac{(1 - 2b)}{(1 - b)}. \quad (51)$$

For $b = 0$ we obtain the usual result of general relativity,

$$\tau_{\pm} \sim \frac{2\pi}{\omega_0} \pm \frac{2\pi a}{c}. \quad (52)$$

From equation (51) we conclude that

$$\tau_+ - \tau_- \sim \frac{4\pi a}{c} \frac{(1 - 2b)}{(1 - b)}. \quad (53)$$

The presence of the factor $(1 - b)^{1/2}$ in the first term of expression (51) suggests that the Newtonian gravitational potential is modified according to

$$\phi(r) = -\frac{GM}{r}(1 - b). \quad (54)$$

This result agrees with that obtained by Wanas in a completely different way [17]. Mashhoon *et al.* [1] considered point like clocks, without internal structure. Such clocks cannot couple with the space-time torsion. An atomic clock certainly has internal structure, and therefore spin.

5 Estimative of the empirical parameter

An estimative of the parameter b can be made by taking into account Einstein's equivalence principle. According to the latter, (a) the trajectory of a freely falling body is independent of its internal structure and composition (known as weak equivalence principle), and (b) the outcome of any local non-gravitational experiment is independent of the velocity of the freely-falling reference frame in which it is performed, and of its position in space and time (local position invariance).

The gravitational redshift of spectral lines is ultimately due to Einstein's equivalence. This effect is universal and independent of the nature of the clock, and is given by

$$\nu = \nu_0 \left(1 + \frac{\phi_N}{c^2} \right), \quad (55)$$

where ν_0 is the proper clock frequency when the $\phi_N = 0$ and ν is the frequency redshifted by the gravitational potential ϕ_N . In the present work the Newtonian gravitational potential is modified by the parameter b . With the help of equation (54) we can write the corresponding result in the context of our analysis,

$$\nu = \nu_0 \left[1 + (1 - b) \frac{\phi_N}{c^2} \right]. \quad (56)$$

Therefore we may determine the parameter b by looking for experiments that violate the equivalence principle.

In the last decades, possible violations of the equivalence principle were tested by means of experiments related to the violation of local position invariance. When the local position invariance principle is violated, the frequency is expected to be [4]

$$\nu = \nu_0 \left[1 + (1 + \beta) \frac{\phi_N}{c^2} \right], \quad (57)$$

β being a dimensionless parameter (positive or negative) that presents a dependence on the internal structure of the clock, and that measures the local position invariance violation of the clock in consideration. Therefore the determination of the parameter b amounts to the fixation of the parameter β .

Experiments with two clocks have been carried out with the purpose of measuring the difference between the two frequencies,

$$(\nu_2 - \nu_1) / \nu_0 = (1 + \beta) \left[\frac{(\phi_{N2} - \phi_{N1})}{c^2} \right], \quad (58)$$

where two identical clocks, 1 and 2, experience different gravitational potentials, ϕ_{N1} , and ϕ_{N2} , respectively. In the Table, we display some of the results of the experiments that determine the parameter β . The Table shows experiments performed with two Cesium atomic clock and two H masers.

Table. Some experiments with two identical clocks to determination of the β parameter [18]

Clocks	β	Reference
H-H	$ \beta_H < 7 \times 10^{-5}$	Vessot <i>et al.</i> [19]
Cs-Cs	$ \beta_{Cs} < 10^{-1}$	Hafele and Keating [20]
Cs-Cs	$ \beta_{Cs} < 1.5 \times 10^{-2}$	Alley [21]
Cs-Cs	$ \beta_{Cs} < 2 \times 10^{-1}$	Briatore and Leschiutta [22]
Cs-Cs	$ \beta_{Cs} < 6 \times 10^{-2}$	Iijima and Fujiwara [23]

The determination of the parameter β_H resulted of the test of redshift based on the measurement of the frequency shift of a H maser on a spacecraft launched upward to 10.000 km compared with a similar maser on Earth. This

is one of the most precise experiments about redshift performed so far. The internal structure of the H maser is simpler than that of the Cesium atomic clock. Taking the values of the parameter $|\beta_H|$ of reference [19] as values of our parameter b , we can calculate the time difference using equation (53). It turns out that there is a difference of 0.007% with respect to Mashhoon's result. The use of $|\beta_{Cs}|$ of reference [21] results in a difference of 1.52% with respect to the expected value.

The parameter b can also be estimated by the redshift experiment conducted by Pound and Snider [24] that measured the frequency shift of gamma-ray photons from ^{57}Fe as the result of Mössbauer effect. In our understanding, the internal structure of ^{57}Fe can couple with the torsion field. The measurement of the redshift yield the value $(0.9990 \pm 0.0076) \times 4.905 \times 10^{-15}$ predicted by the equivalence principle. Our prediction is

$$(1 - b) \times 4.905 \times 10^{-15}, \quad (59)$$

and therefore $b < (0.0010 \pm 0.0076) \approx 7.6 \times 10^{-3}$. This result is in agreement with the results of the Table displayed above for other types of atoms, and in particular with those of Ref. [21]. Such value of b yields a difference of 0.77% with respect to the general relativity prediction.

The equivalence principle can be tested using two nonidentical clocks in the same gravitational potential [18], e.g., the Cs clock and Mg clock. In this experiment, it is measured the difference between the parameters $|\beta_{Cs} - \beta_{Mg}| < 7 \times 10^{-4}$ that represent the coupling of the hyperfine and fine-structure transition in these atoms. This result may indicate the dependence of the fine-structure constant with the gravitational potential. A further experiment with two nonidentical clocks yield $|\beta_{Cs} - \beta_H| < 2.1 \times 10^{-5}$ [25].

6 Weak field approximation

Making use of a Lagrangian quadratic in the torsion tensor, constructed in terms of three free parameters c_1 , c_2 , and c_3 , and considering the weak field approach, Hayashi and Shirafuji [9] wrote down the geodesic equation for a particle in the weak field approximation according to

$$\frac{d^2 x^i}{dt^2} = \frac{2}{9} k \frac{(c_1 + 4c_2)}{c_1 c_2} \frac{\partial}{\partial x^i} \phi_N, \quad i = 1, 2, 3, \quad (60)$$

where $k = c^3/(16\pi G)$. A particular combination of the parameters c_1 , c_2 , and c_3 leads to the condition for the Newtonian limit,

$$c_2 = -\frac{\left(c_1 + \frac{2}{3}k\right)}{\left(1 + \frac{9}{8k}c_1\right)} + \frac{2}{3}k. \quad (61)$$

Wanas *et al.* [6] proposed the particle equation in the weak field limit of the gravitational field to be

$$\frac{d^2 x^i}{dt^2} = -\frac{\partial}{\partial x^i} \phi_s, \quad i = 1, 2, 3, \quad (62)$$

$$\phi_s = (1 - b) \phi_N, \quad (63)$$

where b acquires the values

$$b = \frac{n}{2} \alpha \gamma, \quad n = 0, 1, 2, 3... \quad (64)$$

and $\alpha = 1/137$ is the fine structure constant; γ is an adjustable parameter to be fixed by the experience. According to interpretation of Wanas *et al.*, depending on the value of n we have particles with spin 0, 1, 2, 3 and so on.

We suggest that the interpretation of Hayashi and Shirafuji, and of Wanas *et al.* may be reconciled by writing

$$b = 1 + \frac{2}{9}k \frac{(c_1 + 4c_2)}{c_1 c_2}. \quad (65)$$

We already know that in order to have a well defined Hamiltonian formulation (in the time gauge condition) it is necessary to have two extra conditions on the parameters c_1 , c_2 and c_3 [10],

$$c_1 + c_2 = 0, \quad (66)$$

and

$$c_1 = \frac{4}{9}c_3. \quad (67)$$

From equations (65), (66) and (67) we obtain

$$\begin{aligned}
c_1 &= -\frac{2}{3}k\frac{1}{1-b}, \\
c_2 &= \frac{2}{3}k\frac{1}{1-b}, \\
c_3 &= -\frac{3}{2}k\frac{1}{1-b}.
\end{aligned}$$

For $b = 0$ (i.e., particles without intrinsic spin) we find that the parameters lead to the teleparallel equivalent of general relativity [10], [13]. A nonzero value of b establishes a connection between the TEGR and the geodesic equation that will eventually match with the experiments. Note, however, that according to Hayashi and Shirafuji the experiments do not confirm that c_i have the exact values given above with $b = 0$.

7 Conclusions

In this work we suggest that there is a connection between the following different issues:

- a) the fixation of a global Lorentz symmetry,
- b) the gravitomagnetic effect,
- c) the absolute parallelism of Einstein,
- d) the equivalence principle,
- e) the intrinsic spin of the particles (a quantum aspect) and the space-time torsion (a classical aspect),
- f) the teleparallel equivalent of general relativity and the conditions of Legendre transform that guarantee a well defined time evolution in the Hamiltonian framework, and
- g) Schwinger's time gauge condition.

We concluded that it is possible to have a modified geodesic equation, and investigated it in the context of the gravitomagnetic effect. We also concluded that is possible to describe (on phenomenological grounds) the spin-torsion interaction, by introducing a small correction to the geodesic equation. For macroscopic bodies and particles without spin, this effect does not occur. The small value of b does not invalidate Einstein's general relativity, because the geodesic equation does not depend on Einstein's equations. All results

of the teleparallel equivalent of general relativity remain valid in the limit $b = 0$. The existence of a small correction suggests a small violation of the principle of equivalence that could be determined experimentally. Future space experiments will indicate the correct value of b . The violation of the local position invariance measured by parameter β could be explained by the interaction of the internal composition of the clocks with the torsion field.

It is possible that there exists a relation between the empirical coupling constant b and the contortion tensor in equation (21). The relation between the contortion tensor with the intrinsic spin of the particles (electrons) is suggested in Hayashi and Shirafuji's work [9].

It remains to discover the variational principle that leads to the correct geodesic equation. Also, it is necessary a further understanding of the coupling between the contortion tensor and the spin of matter. Efforts in this respect will be carried out.

Acknowledgements

One of us, A. A. Sousa, is grateful to Faculdades Planalto for financial support.

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